5.4

Quadratic Expressions

Learning Objectives:

- To discuss the sign of a quadratic expression and to study the change in signs
- To find extreme values of quadratic expressions

  AND
- To practice the related problems

A polynomial of the form $ax^2 + bx + c$, where a, b, c are real or complex numbers and $a \neq 0$, is called a ***quadratic expression in*** $\boldsymbol{x}$.

Throughout this module we consider quadratic expressions with real coefficients. In this module we discuss the sign of a quadratic expression, its change in signs and maximum and minimum values.

Sign of a quadratic expression:

Theorem 1: Let $a, b, c \in \boldsymbol{R},\ a \neq 0$, then

(i) The roots of $ax^2 + bx + c = 0$ are non-real complex numbers if and only if the quadratic expression $ax^2 + bx + c$ and a have the same sign for all $x \in \boldsymbol{R}$.

(ii) If $ax^2 + bx + c = 0$ has equal roots then the quadratic expression $ax^2 + bx + c$ and a have the same sign for all $x \in \boldsymbol{R},\ x \neq -\frac{b}{2a}$.

Proof:

We have $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$$

Thus, $\frac{ax^2+bx+c}{a} = \left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}$ --- (1)

(i) If the roots of $ax^2 + bx + c = 0$ are non-real complex numbers, then $b^2 - 4ac < 0$, i.e., $4ac - b^2 > 0$ and from (1) $\frac{ax^2+bx+c}{a} > 0$ for all $x \in \boldsymbol{R}$

$\Rightarrow ax^2 + bx + c$ and a have the same sign for all $x \in \boldsymbol{R}$.

Conversely, suppose that $ax^2 + bx + c$ and a have the same sign for all $x \in \boldsymbol{R}$.

$\Rightarrow \frac{ax^2+bx+c}{a} > 0 \Rightarrow \left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} > 0$, $\forall x \in \boldsymbol{R}$

Taking $x = -\frac{b}{2a}$, we obtain

$$\frac{4ac-b^2}{4a^2} > 0 \Rightarrow 4ac - b^2 > 0 \Rightarrow b^2 - 4ac < 0$$

Hence the roots of $ax^2 + bx + c = 0$ are non-real complex numbers.

This proves the first part of the theorem.

(ii) If the equation $ax^2 + bx + c = 0$ has equal roots, then $b^2 - 4ac = 0$ and from (1)

$$\frac{ax^2+bx+c}{a} = \left(x + \frac{b}{2a}\right)^2 > 0 \text{ for all } x \in \boldsymbol{R}, x \neq -\frac{b}{2a}$$

Thus, $ax^2 + bx + c = 0$ and a have the same sign for all $x \in \boldsymbol{R}, x \neq -\frac{b}{2a}$.

This proves the second part of the theorem.

Example1:

Determine the sign of the quadratic expression $x^2 - x + 2$ for $x \in \boldsymbol{R}$.

Solution:

The discriminant $= (-1)^2 - 4 \cdot 1 \cdot 2 = -7 < 0$. Therefore, the roots of the quadratic equation $x^2 - x + 2 = 0$ are non-real complex numbers.

Therefore, $x^2 - x + 2$ and the coefficient of x^2 have the same sign for all $x \in \boldsymbol{R}$. Since the coefficient of x^2 is $1 > 0$,
$x^2 - x + 2 > 0\ ,\ \forall\, x \in \boldsymbol{R}$.

Change in signs of a quadratic expression:

Theorem 2: Let α and β be real roots of $ax^2 + bx + c = 0$ and $\alpha < \beta$. Then

(i) If $x \in (\alpha, \beta)$, then $ax^2 + bx + c$ and a have opposite signs.

(ii) If $x \in (-\infty, \alpha) \cup (\beta, \infty)$, then $ax^2 + bx + c$ and a have the same sign.

Proof:

Since α, β are the roots of $ax^2 + bx + c = 0$;

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

Therefore, $\frac{ax^2+bx+c}{a} = (x - \alpha)(x - \beta)$

(i) Suppose $x \in (\alpha, \beta)$. Then $x - \alpha > 0, x - \beta < 0$ and

$$\frac{ax^2+bx+c}{a} = (x - \alpha)(x - \beta) < 0$$

Thus $ax^2 + bx + c$ and a have opposite signs.

(ii) Suppose $x \in (-\infty, \alpha) \cup (\beta, \infty)$. Then $x \in (-\infty, \alpha)$ or $x \in (\beta, \infty)$.

a) If $x \in (-\infty, \alpha)$, then $x < \alpha < \beta$ and

$$x - \alpha < 0, x - \beta < 0.$$

Therefore, $\frac{ax^2+bx+c}{a} = (x-\alpha)(x-\beta) > 0$

Thus $ax^2 + bx + c$ and a have the same sign.

b) If $x \in (\beta, \infty)$, then $\alpha < \beta < x$ and

$$x - \alpha > 0, x - \beta > 0.$$

Therefore, $\frac{ax^2+bx+c}{a} = (x-\alpha)(x-\beta) > 0$

Thus $ax^2 + bx + c$ and a have the same sign.

Combining the above two cases, the second part of the theorem follows.

Hence the theorem.

Example2:

Discuss the sign of the quadratic expression $4x - 5x^2 + 2$ where $x \in \boldsymbol{R}$.

Solution:

We have, $-5x^2 + 4x + 2$.

It's discriminant is $b^2 - 4ac = 16 - 4 \cdot (-5) \cdot 2 = 56 > 0$.

The roots are real and they are $\alpha = \frac{2-\sqrt{14}}{5}$, $\beta = \frac{2+\sqrt{14}}{5}$.

Now $-5x^2 + 4x + 2$ and -5, the coefficient of x^2 have opposite signs if $x \in (\alpha, \beta)$ and have the same sign if $x \in (-\infty, \alpha) \cup (\beta, \infty)$.

Thus $4x - 5x^2 + 2 > 0$ if $x \in (\alpha, \beta)$

and $4x - 5x^2 + 2 < 0$ if $x \in (-\infty, \alpha) \cup (\beta, \infty)$.

Extreme values of a quadratic expression:

We show that the extreme values of a quadratic expression with real coefficients depend on the sign of its leading coefficient.

Theorem 3:

Let $a, b, c \in \boldsymbol{R}, a \neq 0$ and $f(x) = ax^2 + bx + c$.

(i) If $a > 0$, then $f(x)$ has absolute minimum at $x = -\frac{b}{2a}$ and the minimum value is $\frac{4ac-b^2}{4a}$;

(ii) If $a < 0$, then $f(x)$ has absolute maximum at $x = -\frac{b}{2a}$ and the maximum value is $\frac{4ac-b^2}{4a}$.

Proof:

We have $f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$ --- (1)

(i) If $a > 0$, then $f(x) \geq \frac{4ac-b^2}{4a}$, $\forall x \in \boldsymbol{R}$ and

$f(x) = \frac{4ac-b^2}{4a}$ when $x = -\frac{b}{2a}$.

This shows that $f(x)$ has absolute minimum at $x = -\frac{b}{2a}$ when $a > 0$ and its minimum value is $\frac{4ac-b^2}{4a}$.

(ii) If $a < 0$, then $f(x) \leq \frac{4ac-b^2}{4a}$, $\forall x \in \boldsymbol{R}$ and

$f(x) = \frac{4ac-b^2}{4a}$ when $x = -\frac{b}{2a}$.

This shows that $f(x)$ has absolute maximum at $x = -\frac{b}{2a}$ when $a < 0$ and its maximum value is $\frac{4ac-b^2}{4a}$.

Hence the theorem.

Example3:

Find the maximum and minimum values of the quadratic expressions

(a) $12x - x^2 - 32$

(b) $ax^2 + bx + a\,;\, a, b \in \boldsymbol{R}, a \neq 0.$

Solution:

(a) Let $f(x) = 12x - x^2 - 32$. The coefficient of x^2 negative.

Therefore, $f(x)$ has absolute maximum at

$x = -\frac{b}{2a} = -\frac{12}{2(-1)} = 6$ and the maximum value is

$f(6) = \frac{4ac-b^2}{4a} = \frac{4(-1)(-32)-12^2}{4(-1)} = 4$.

(b) Let $f(x) = ax^2 + bx + a\,;\; a, b \in \boldsymbol{R}, a \neq 0.$

If $a > 0$, then $f(x)$ has absolute minimum at $x = -\frac{b}{2a}$ and

the minimum value is $\frac{4ac-b^2}{4a} = \frac{4a^2-b^2}{4a}$.

If $a < 0$, then $f(x)$ has absolute maximum at $x = -\frac{b}{2a}$

and the maximum value is $\frac{4ac-b^2}{4a} = \frac{4a^2-b^2}{4a}$.

P1:

Determine the sign of the quadratic expression $x^2 - 8x + 16$.

Solution:

We have, $x^2 - 8x + 16$.

The discriminant $= (-8)^2 - 4 \cdot 1 \cdot 16 = 0$.

The roots of the quadratic equation $x^2 - 8x + 16 = 0$ are real and equal.

Therefore, by the second part of the Theorem 1, $x^2 - 8x + 16$ and the coeff.of x^2 have the same sign for all $x \in \boldsymbol{R}, x \neq -\frac{b}{2a} \neq 4$. Since the coefficient of x^2 is $1 > 0$, $x^2 - 8x + 16 > 0$, $\forall\, x \in \boldsymbol{R}, x \neq 4$.

P2:

For what values of x, the expression $2x^2 + 5x - 3$ is negative.

Solution:

We have, $2x^2 + 5x - 3$.

It's discriminant is $b^2 - 4ac = 25 - 4 \cdot 2 \cdot (-3) = 1 > 0$.

The roots are real and they are $\alpha = -3\,, \beta = \frac{1}{2}$.

Now, by Theorem 2, $2x^2 + 5x - 3$ and the coefficient of x^2 have opposite signs if $x \in (\alpha, \beta) = \left(-3\,, \frac{1}{2}\right)$ and have the same sign if $x \in (-\infty, \alpha) \cup (\beta, \infty) = (-\infty, -3) \cup \left(\frac{1}{2}, \infty\right)$.

Since the coefficient of x^2 is $2 > 0$, $2x^2 + 5x - 3 < 0$ if $x \in \left(-3\,, \frac{1}{2}\right)$ and

$2x^2 + 5x - 3 > 0$ if $x \in (-\infty, -3) \cup \left(\frac{1}{2}, \infty\right)$.

Therefore, the expression $2x^2 + 5x - 3$ is negative for $x \in \left(-3\,, \frac{1}{2}\right)$.

P3:

Find the maximum or minimum of the expression $x^2 + 5x + 6$.

Solution:

We have, x^2+5x+6. Comparing this expression with ax^2+bx+c, we have $a=1, b=5, c=6$. Since $a=1>0$, (by the first part of Theorem 3), x^2+5x+6 has absolute minimum at

$$x=-\frac{b}{2a}=-\frac{5}{2\cdot 1}=-\frac{5}{2}$$

and the minimum value is $\frac{4ac-b^2}{4a}=\frac{4(1)(6)-(5)^2}{4(1)}=-1$.

Therefore, the given expression has the minimum value -1 at $x=-\frac{5}{2}$.

P4:

Find the maximum or minimum of the expression $2x - 7 - 5x^2$

Solution:

We have, $-5x^2 + 2x - 7$. Comparing this expression with $ax^2 + bx + c$, we have $a = -5, b = 2, c = -7$. Since $a = -5 < 0$, (by the second part of Theorem 3), $-5x^2 + 2x - 7$ has absolute maximum at

$$x = -\frac{b}{2a} = -\frac{2}{2(-5)} = \frac{1}{5}$$

and the maximum value is $\frac{4ac-b^2}{4a} = \frac{4(-5)(-7)-(2)^2}{4(-5)} = -\frac{34}{5}$.

Therefore, the given expression has the maximum value $-\frac{34}{5}$ at $x = \frac{1}{5}$.

IP1:

Determine the sign of the quadratic expression $x^2 + x + 1$.

Solution:

We have, $x^2 + x + 1$.

The discriminant $= (1)^2 - 4 \cdot 1 \cdot 1 = -3 < 0$.

The roots of the quadratic equation $x^2 + x + 1 = 0$ are non-real complex numbers.

Therefore, by the first part of the Theorem 1, $x^2 + x + 1$ and the coeff.of x^2 have the same sign for all $x \in \boldsymbol{R}$. Since the coefficient of x^2 is $1 > 0$, $x^2 + x + 1 > 0, \ \forall\, x \in \boldsymbol{R}$.

IP2:

For what values of x, the expression $x^2 - 5x + 6$ is positive.

Solution:

We have, $x^2 - 5x + 6$.

It's discriminant is $b^2 - 4ac = 25 - 4 \cdot 1 \cdot 6 = 1 > 0$.

The roots are real and they are $\alpha = 2, \beta = 3$.

Now, by Theorem 2, $x^2 - 5x + 6$ and the coefficient of x^2 have opposite signs if $x \epsilon (\alpha, \beta) = (2, 3)$ and have the same sign if $x \epsilon (-\infty, \alpha) \cup (\beta, \infty) = (-\infty, 2) \cup (3, \infty)$.

Since the coefficient of x^2 is $1 > 0$, $x^2 - 5x + 6 < 0$ if $x \epsilon (2,3)$ and $x^2 - 5x + 6 > 0$ if $x \epsilon (-\infty, 2) \cup (3, \infty)$.

Therefore, the expression $x^2 - 5x + 6$ is positive for $x \epsilon (-\infty, 2) \cup (3, \infty)$.

IP3:

Find the maximum or minimum of the expression $x^2 - 8x + 17$

Solution:

We have, $x^2 - 8x + 17$. Comparing this expression with $ax^2 + bx + c$, we have $a = 1, b = -8, c = 17$. Since $a = 1 > 0$, (by the first part of Theorem 3), $x^2 - 8x + 17$ has absolute minimum at

$$x = -\frac{b}{2a} = -\frac{(-8)}{2 \cdot 1} = 4$$

and the minimum value is $\frac{4ac - b^2}{4a} = \frac{4(1)(17) - (-8)^2}{4(1)} = 1.$

Therefore, the given expression has the minimum value 1 at $x = 4$.

IP4:

Find the maximum or minimum of the expression $2x - x^2 + 7$.

Solution:

We have, $-x^2 + 2x + 7$. Comparing this expression with $ax^2 + bx + c$, we have $a = -1, b = 2, c = 7$. Since $a = -1 < 0$, (by the second part of Theorem 3), $-x^2 + 2x + 7$ has absolute maximum at

$$x = -\frac{b}{2a} = -\frac{2}{2(-1)} = 1$$

and the maximum value is $\frac{4ac-b^2}{4a} = \frac{4(-1)(7)-(2)^2}{4(-1)} = 7$.

Therefore, the given expression has the maximum value 7 at $x = 1$.

1. Determine the sign of the following expressions for $x \in \boldsymbol{R}$.

   a. $x^2 - 5x + 6$

   b. $x^2 - 5x + 4$

   c. $x^2 - x + 3$

2. For what values of x, the following expressions are positive?

a. $3x^2 + 4x + 4$

b. $4x - 5x^2 + 2$

c. $4x - 5x^2 + 1$

d. $x^2 - 5x + 14$

3. For what values of x, the following expressions are negative?

a. $x^2 - 7x + 10$

b. $15 + 4x - 3x^2$

c. $x^2 - 5x - 6$

d. $2x - 3 - 6x^2$

e. $-7x^2 + 8x - 9$

4. Find the maximum or minimum of the following expressions as x varies over $\boldsymbol{R}$.

a. $2x - 7 - 5x^2$

b. $3x^2 + 2x + 11$

c. $x^2 - x + 7$

d. $2x + 5 - 3x^2$